

# GLOBAL CROSS SECTIONS FOR ANOSOV FLOWS

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ABSTRACT. We provide a new criterion for the existence of a global cross section to a volume-preserving Anosov flow. The criterion is expressed in terms of expansion and contraction rates of the flow and is more general than the previous results of similar kind.

## 1. INTRODUCTION

Henri Poincaré introduced the idea of a cross section to a flow to study the 3-body problem. A global cross section to a flow  $\Phi$  on a manifold  $M$  is a codimension one submanifold  $\Sigma$  of  $M$  such that  $\Sigma$  intersects every orbit of  $\Phi$  transversely. It is natural to ask whether any given non-singular flow admits one.

If  $\Sigma$  is a global cross section for  $\Phi$ , it is not hard to check that every orbit which starts on  $\Sigma$  returns to  $\Sigma$  after some positive time, defining the Poincaré first-return map  $g : \Sigma \rightarrow \Sigma$ . The analysis of  $\Phi$  can then be reduced to the study of the map  $g$ , which in principle can be an easier task. The flow can be reconstructed from the Poincaré map by suspending it (cf., [KH95]).

The object of this paper is to investigate the existence of global cross sections to volume-preserving Anosov flows.

Recall that a non-singular flow  $\Phi = \{f_t\}$  on a closed (compact and without boundary) Riemannian manifold  $M$  is called **Anosov** if there exists an invariant splitting  $TM = E^{ss} \oplus E^c \oplus E^{uu}$  of the tangent bundle of  $M$  and uniform constants  $c > 0$ ,  $0 < \mu_- \leq \mu_+ < 1$  and  $\lambda_+ \geq \lambda_- > 1$  such that the center bundle  $E^c$  is spanned by the infinitesimal generator  $X$  of the flow and for all  $v \in E^{ss}$ ,  $w \in E^{uu}$ , and  $t \geq 0$ , we have

$$\frac{1}{c} \mu_-^t \|v\| \leq \|Tf_t(v)\| \leq c \mu_+^t \|v\|, \quad (1.1)$$

and

$$\frac{1}{c} \lambda_-^t \|w\| \leq \|Tf_t(w)\| \leq c \lambda_+^t \|w\|, \quad (1.2)$$

where  $Tf_t$  denotes the derivative (or tangent map) of  $f_t$ . We call  $E^{ss}$  and  $E^{uu}$  the **strong stable** and **strong unstable bundles**;  $E^{cs} = E^c \oplus E^{ss}$  and  $E^{cu} = E^c \oplus E^{uu}$  are called the **center stable** and **center unstable bundles**. It is well-known [HPS77, Has94] that all of them are Hölder continuous and uniquely integrable [Ano67]. The corresponding foliations will be denoted by  $W^{ss}, W^{uu}, W^{cs}$ , and  $W^{cu}$ . They are also Hölder continuous in the sense that each one admits Hölder foliation charts. This means that if  $W^\sigma$  ( $\sigma \in \{ss, uu, cs, cu\}$ ) is  $C^\theta$ , then every point in  $M$  lies in a  $C^\theta$  chart  $(U, \varphi)$  such that in  $U$  the local  $W^\sigma$ -leaves are given by  $\varphi_{k+1} = \text{constant}, \dots, \varphi_n = \text{constant}$ , where  $\varphi = (\varphi_1, \dots, \varphi_n)$  is a  $C^\theta$  homeomorphism and  $k$  is the dimension of  $W^\sigma$ . The leaves of all invariant foliations are as smooth as the flow. See also [PSW97] for a discussion of regularity of Hölder foliations.

**Related work.** The first results on the existence of global cross sections to Anosov flows were proved by Plante in [Pla72]. He showed that if  $E^{ss} \oplus E^{uu}$  is a uniquely integrable distribution or equivalently, if the foliations  $W^{ss}$  and  $W^{uu}$  are jointly integrable<sup>1</sup>, then the Anosov flow admits a global cross section. Sharp [Sha93] showed that a transitive Anosov flow admits a global cross section if it is not homologically full; this means that for every homology class  $\alpha \in H_1(M, \mathbb{Z})$  there is a closed  $\Phi$ -orbit  $\gamma$  whose homology class equals  $\alpha$ . (This is equivalent to the condition that there is *no* fully supported  $\Phi$ -invariant ergodic probability measure whose asymptotic cycle in the sense of Schwartzman [Sch57] is trivial.) Along different lines Bonatti and Guelman [BG09] showed that if the time-one map of an Anosov flow can be  $C^1$  approximated by Axiom A diffeomorphisms, then the flow is topologically equivalent to the suspension of an Anosov diffeomorphism.

Let  $n_s = \dim E^{ss}$  and  $n_u = \dim E^{uu}$ . If  $n_u = 1$  or  $n_s = 1$ , the Anosov flow is said to be of codimension one. In the discussion that follows we always assume  $n_u = 1$ . In [Ghy89] Ghys proved the existence of global cross sections for codimension one Anosov flows in the following cases: (1) if  $E^{su} = E^{ss} \oplus E^{uu}$  is  $C^1$  and  $n \geq 4$  (in this case the global cross section has constant return time); if (2) the flow is volume-preserving,  $n \geq 4$  and  $W^{cs}$  is of class  $C^2$ . This was generalized by the author in [Sim96] and [Sim97] where we showed that a codimension one Anosov flow admits a global cross section if any of the following assumptions is satisfied: (1)  $E^{su}$  is Lipschitz (in the sense that it is locally spanned by Lipschitz vector fields) and  $n \geq 4$ ; (2) the flow is volume-preserving,  $n \geq 4$ , and  $E^{su}$  is  $C^\theta$ -Hölder for *all*  $\theta < 1$  (3) the flow is volume-preserving,  $n \geq 4$ , and  $E^{cs}$  is of class  $C^{1+\theta}$  for *all*  $\theta < 1$ . Note that all the regularity assumptions above require that the invariant bundles be smoother than they usually are:  $E^{su}$  is generically only Hölder continuous and in the codimension one case,  $E^{cs}$  is generically only  $C^{1+\theta}$  for some small  $0 < \theta < 1$ . See [Has94, Has97] and [HW99].

The goal of this paper is to establish the following result.

**Theorem.** *Let  $\Phi = \{f_t\}$  be a volume-preserving Anosov flow on a closed Riemannian manifold  $M$  and let  $0 < \theta \leq 1$  be the smaller of the Hölder exponents of  $W^{ss}$  and  $W^{cs}$ . If*

$$\mu_+^{(n_s-1)\theta} \lambda_+^{(n_u-1)\theta} < \mu_-^{2(1-\theta)}, \quad (1.3)$$

*then  $\Phi$  admits a global cross section.*

**Remarks.** (a) The condition (1.3) has a chance of being satisfied only if  $n_u$  is much smaller than  $n_s$ . If  $n_u > n_s$ , then by reversing time it is easy to show that

$$\lambda_+^{2(1-\alpha)} < \lambda_-^{(n_u-1)\alpha} \mu_-^{(n_s-1)\alpha}$$

also implies the existence of a global cross section, where  $\alpha$  is the minimum of the Hölder exponents of  $E^{uu}$  and  $E^{cu}$ .

(b) If the flow is of codimension one with  $n_u = 1$ , then (1.3) reduces to

$$\mu_+^{(n-3)\theta} < \mu_-^{2(1-\theta)}. \quad (1.4)$$

It is well-known (cf., [Has94] and [HPS77]) that the center stable bundle  $E^{cs}$  and strong unstable bundle  $E^{uu}$  of a volume-preserving Anosov flow in dimensions  $n \geq 4$  are both  $C^{1+\text{Hölder}}$ . Thus if  $E^{su}$  is Lipschitz as in [Sim96] or  $C^\theta$ , for all  $\theta < 1$ , as in [Sim97], then (1.4) is clearly satisfied. If  $E^{cs}$  is  $C^{1+\theta}$  for all  $\theta < 1$  as in [Sim97], then it is not hard to show that  $E^{ss}$  is necessarily of class  $C^\theta$  for all  $\theta < 1$ , which again implies (1.4). Therefore, in the case of volume-preserving codimension one Anosov flows, our result implies all the previously known criteria for the existence of global cross sections.

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<sup>1</sup>This means that locally speaking the  $W^{uu}$ -holonomy between local  $W^{cs}$ -leaves takes local  $W^{ss}$ -leaves to  $W^{ss}$ -leaves.

- (c) In the early 1970's, prompted by a dearth of examples, A. Verjovsky conjectured [Ver74] that every codimension one Anosov flow in dimensions  $n \geq 4$  admits a global cross section. The importance of Verjovsky's conjecture stems from the fact that codimension one Anosov diffeomorphisms were classified by Franks [Fra70] and Newhouse [New70] who showed that every such diffeomorphism is topologically conjugate to a linear hyperbolic automorphism of a torus. Therefore, the affirmation of Verjovsky's conjecture would yield a complete classification of codimension one Anosov flows in dimensions  $n \geq 4$ .

Progress towards Verjovsky's conjecture was made in the early 1980's by Plante [Pla81, Pla83] and Armendariz [Arm82] who showed that the conjecture holds if the fundamental group of the manifold is solvable. By the work of Asaoka [Asa08] it follows that it suffices to prove the conjecture for volume-preserving flows, since any topologically transitive codimension one Anosov flow is topologically equivalent to a volume-preserving one. As of this writing, the conjecture remains open.

Throughout this paper smooth will mean of class  $C^\infty$ .

**Outline of the proof.** The main idea of the proof of the theorem is to find a smooth closed 1-form  $\eta$  such that  $\eta(X) > 0$ , where  $X$  is the infinitesimal generator of the Anosov flow. It is not hard to see that this immediately implies the existence of a global cross section (cf., §3). To construct  $\eta$ , we use the fact that for any  $k$ -form  $\xi$ , the  $C^0$ -distance from  $\xi$  to the space of closed  $k$ -forms is bounded above by the  $C^0$  norm  $\|d\xi\|$ ; see Proposition 2.4. It therefore suffices to construct a smooth 1-form  $\xi$  such that  $|\xi(X_p)| > \|d\xi\|$ , for all  $p \in M$ , since Proposition 2.4 then yields a smooth closed 1-form  $\eta$  such that  $\eta(X) > 0$ . We will actually construct a smooth 1-form  $\xi$  such that  $\xi(X) = 1$  and  $\|d\xi\| < 1$ .

The construction of  $\xi$  is divided into two steps. In the first step, we find an initial candidate for  $\xi$  such that the norm of its exterior derivative restricted to  $E^{cs}$  is small, while its total norm blows up in a way controlled by the Hölder exponent  $\theta$ . More precisely, for each  $\varepsilon > 0$  we construct a smooth 1-form  $\xi_0^\varepsilon$  on  $M$  such that  $\xi_0^\varepsilon(X) = 1$ ,  $\|d\xi_0^\varepsilon \upharpoonright_{E^{cs}}\| \leq D\varepsilon^\theta$ , and  $\|d\xi_0^\varepsilon\| \leq K\varepsilon^{\theta-1}$ , where  $D$  and  $K$  are positive constants independent of  $\varepsilon$ . This is achieved by carefully building smooth local cross sections and the corresponding flow boxes; cf., §2.2.

In the second step, we pull back  $\xi_0^\varepsilon$  by  $f_{-t}$  for suitable  $t > 0$  to make the norm of its exterior derivative small in the remaining directions. For this, we use an estimate (see Lemma 2.5) on the growth of  $\|Tf_{-t}(v \wedge w)\|$ , for  $t > 0$ ,  $v \in E^{ss}$  and  $w \in E^{uu}$ . Assuming (1.3), we then show that there exist  $\varepsilon > 0$  and  $t > 0$  such that  $\xi = f_{-t}^* \xi_0^\varepsilon$  has the desired properties.

## 2. PRELIMINARIES

**2.1. Regularization.** Here we recall a standard technique for approximating locally integrable functions by smooth ones called **regularization** or **mollification** (see [Eva98, Ste70]). Define the standard mollifier  $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$\eta(x) = \begin{cases} A_0 \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where  $A_0$  is chosen so that  $\int \eta \, dx = 1$ . For every  $\varepsilon > 0$ , set  $\eta_\varepsilon(x) = \varepsilon^{-n} \eta(x/\varepsilon)$ . Note that the support of  $\eta_\varepsilon$  is contained in the ball  $B(0, \varepsilon)$  of radius  $\varepsilon$  centered at the origin and that  $\int \eta_\varepsilon \, dx = 1$ .

For a locally integrable function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  define

$$u^\varepsilon(x) = (u * \eta_\varepsilon)(x) = \int_{\mathbb{R}^n} u(y) \eta_\varepsilon(x - y) dy = \int_{\mathbb{R}^n} u(x - y) \eta_\varepsilon(y) dy.$$

**2.1. Proposition.** *Assume that  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  is locally integrable. Then:*

- (a)  $u^\varepsilon \in C^\infty(\mathbb{R}^n)$ .
- (b) If  $u \in L^\infty$ , then  $\|u^\varepsilon\|_{L^\infty} \leq \|u\|_{L^\infty}$ .
- (c) If  $u$  is continuous, then  $u^\varepsilon \rightarrow u$ , uniformly as  $\varepsilon \rightarrow 0$ . If  $u \in C^\theta$  ( $0 < \theta \leq 1$ ), then  $\|u^\varepsilon - u\|_{C^0} \leq \|u\|_{C^\theta} \varepsilon^\theta$ , where the  $C^\theta$ -norm is the sum of its sup-norm and its best Hölder constant.
- (d) If  $u \in C^\theta$ , then

$$\|du^\varepsilon\| \leq \|d\eta\|_{L^1} \|u\|_{C^\theta} \varepsilon^{\theta-1}$$

where  $\|d\eta\|_{L^1} = \max_i \int_{\mathbb{R}^n} |\partial \eta / \partial x_i| dx$  and  $\|du^\varepsilon\|$  denotes the maximum of the sup-norms of the partial derivatives of  $u^\varepsilon$ .

*Proof.* Proofs of (a), (b) and the first part of (c) can be found in [Eva98]. For the second part of (c), we have

$$\begin{aligned} |u^\varepsilon(x) - u(x)| &= \left| \int_{B(0,\varepsilon)} \eta_\varepsilon(y) [u(x - y) - u(x)] dy \right| \\ &\leq \|u\| \varepsilon^\theta \int_{B(0,\varepsilon)} \eta_\varepsilon(y) dy \\ &= \|u\|_{C^\theta} \varepsilon^\theta. \end{aligned}$$

If  $u \in C^1$ , then the same estimates hold with  $\theta$  replaced by 1.

Observe that since  $\eta_\varepsilon$  has compact support,

$$\int_{\mathbb{R}^n} \frac{\partial \eta_\varepsilon}{\partial x_i}(y) dy = 0, \tag{2.1}$$

for  $1 \leq i \leq n$ . Note also that

$$\frac{\partial \eta_\varepsilon}{\partial x_i}(x) = \frac{1}{\varepsilon^{n+1}} \frac{\partial \eta}{\partial x_i} \left( \frac{x}{\varepsilon} \right).$$

Assuming  $u \in C^\theta$ , we obtain (d):

$$\begin{aligned} \left| \frac{\partial u^\varepsilon}{\partial x_i}(x) \right| &= \left| \int_{\mathbb{R}^n} u(x - y) \frac{\partial \eta_\varepsilon}{\partial x_i}(y) dy \right| \\ &\stackrel{\text{by (2.1)}}{=} \left| \int_{B(0,\varepsilon)} [u(x - y) - u(x)] \frac{\partial \eta_\varepsilon}{\partial x_i}(y) dy \right|, \\ &\leq \|u\|_{C^\theta} \varepsilon^\theta \int_{B(0,\varepsilon)} \left| \frac{\partial \eta_\varepsilon}{\partial x_i}(y) \right| dy \\ &= \|u\|_{C^\theta} \varepsilon^\theta \int_{B(0,\varepsilon)} \frac{1}{\varepsilon^{n+1}} \left| \frac{\partial \eta}{\partial x_i} \left( \frac{y}{\varepsilon} \right) \right| dy \\ &\stackrel{z=\frac{y}{\varepsilon}}{=} \|u\|_{C^\theta} \varepsilon^\theta \cdot \frac{1}{\varepsilon} \int_{B(0,1)} \left| \frac{\partial \eta}{\partial x_i}(z) \right| dz, \\ &\leq \|d\eta\|_{L^1} \|u\|_{C^\theta} \varepsilon^{\theta-1}. \end{aligned}$$

□

**2.2. Corollary.** *Let  $M$  be a compact manifold without boundary. Fix a finite atlas  $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$  of  $M$ . If  $u : U \rightarrow \mathbb{R}$  is  $C^\theta$  and  $U \subset U_j$  for some  $j \in I$ , then there exist  $\varepsilon_* > 0$  and a family of smooth approximations  $u^\varepsilon$  ( $0 < \varepsilon < \varepsilon_*$ ) of  $u$  such that:*

- (a)  $u^\varepsilon$  is defined on  $U^\varepsilon \subset U$  where  $U^\varepsilon = \varphi_j^{-1}\{x \in \varphi_j(U) : d(x, \partial\varphi_j(U)) > \varepsilon\}$ .
- (b)  $\|u^\varepsilon - u\|_{C^0} \leq \kappa\varepsilon^\theta$ .
- (c)  $\|du^\varepsilon\|_{C^0} \leq \kappa\varepsilon^{\theta-1}$ .

Here  $\kappa > 0$  depends only on  $\mathcal{A}$  and  $\|u\|_{C^\theta}$ .

*Proof.* The family  $u^\varepsilon = [(u \circ \varphi_j^{-1}) * \eta_\varepsilon] \circ \varphi_j$  has the desired properties. We can take  $\varepsilon_*$  to be any positive number such that for  $\varepsilon < \varepsilon_*$ , the sets  $U^\varepsilon$  defined above are non-empty.  $\square$

**2.2. Construction of local cross sections.** In this section we construct a finite covering of the manifold by smooth flow boxes defined by carefully chosen smooth local cross sections.

Let  $p \in M$  be arbitrary and choose a  $W^{ss}$ -foliation chart  $(U, \varphi)$  containing  $p$ . This means that  $\varphi : U \rightarrow \mathbb{R}^n$  is a  $C^\theta$ -homeomorphism such that the local  $W^{ss}$ -leaves in  $U$  are given by

$$\varphi_{n_s+1} = \text{constant}, \dots, \varphi_n = \text{constant},$$

where  $\varphi = (\varphi_1, \dots, \varphi_n)$ . We can also arrange that the local  $W^{cs}$ -leaves in  $U$  are defined by

$$\varphi_{n_s+1} = \text{constant}, \dots, \varphi_{n-1} = \text{constant},$$

so that in each local  $W^{cs}$ -leaf in  $U$   $W^{ss}$  is given by  $\varphi_n = \text{constant}$ . Furthermore, we can take  $U$  so that its closure is contained in a *smooth* chart for  $M$ . This condition will allow us to mollify continuous functions defined on  $U$  without having to shrink the domain.

Even though  $\varphi$  is only Hölder, the flow invariance of  $W^{ss}$  implies that each  $\varphi_i$  is differentiable with respect to  $X$ . Since  $X$  is tangent to the  $W^{cs}$  leaves, it follows that  $X\varphi_i = 0$ , for  $n_s + 1 \leq i \leq n - 1$ . Furthermore, the restriction of  $W^{ss}$  to  $W^{cs}$ -leaves is as smooth as the flow, so  $\varphi_n$  is smooth on the local  $W^{cs}$ -leaves in  $U$ . Since  $X$  is uniformly transverse to  $W^{ss}$  it is clear that  $X\varphi_n \neq 0$  and by continuity there exists  $\delta > 0$  such that  $|X\varphi_n| \geq \delta$  on  $U$ .

Define

$$\Sigma_0 = \bigcup_{x \in W_{\text{loc}}^{uu}(p)} W_{\text{loc}}^{ss}(x),$$

where  $W_{\text{loc}}^\sigma(x)$  denotes the local  $W^\sigma$ -leaf in  $U$  (for  $\sigma \in \{uu, ss\}$ ). Then  $\Sigma_0$  is a Hölder continuous local cross section for the flow. (This makes sense, since the intersection of  $\Sigma_0$  with each local  $W^{cs}$ -leaf is a local  $W^{ss}$ -leaf, hence smooth and transverse to the flow.) Let  $\Sigma$  be a slightly smaller compact subset of  $\Sigma_0$  containing  $p$ .

We claim that there exists  $T > 0$ , depending on  $\Sigma$  and  $U$ , such that

$$V \stackrel{\text{def}}{=} \bigcup_{|t| < T} f_t(\Sigma)$$

is a Hölder continuous flow box contained in  $U$ . Assume the contrary; it follows that the map  $(t, q) \mapsto f_t(q)$  fails to be 1-1 on  $(-T, T) \times \Sigma$ , for any  $T > 0$ . Thus there exist sequences  $(q_k)$  and  $(T_k)$  such that  $q_k \in \Sigma$ ,  $T_k > 0$ ,  $T_k \rightarrow 0$  and  $f_{T_k}(q_k) \in \Sigma$ , for all  $k$ . Since  $f_{T_k}(q_k)$  and  $q_k$  lie in the same local  $W^{cs}$ -leaf, it follows that  $\varphi_n(f_{T_k}(q_k)) = \varphi_n(q_k)$ . On the other hand, by compactness of  $\Sigma$ ,  $(q_k)$  has a subsequence  $(q_{k_j})$  which converges to some  $q \in \Sigma_0$ . This implies  $X\varphi_n(q) = 0$ , which contradicts  $|X\varphi_n| \geq \delta$ . Therefore,  $T$  exists; we will call it the *length* of the continuous flow box  $V$  (although a more appropriate name would be half-length).

Define  $\tau : V \rightarrow \mathbb{R}$  by

$$\tau(f_t q) = t,$$

for  $q \in \Sigma$  and  $|t| < T$ . It is clear that  $\tau$  is  $C^\theta$ ,  $X\tau = 1$  and  $\tau$  is constant on the local  $W^{ss}$ -leaves in  $V$ .

Next we approximate  $\Sigma, \tau$  and  $V$  by smooth objects with similar properties.

**2.3. Lemma.** *There exists  $\varepsilon_* > 0$  such that for every  $0 < \varepsilon < \varepsilon_*$  there exist an open set  $V^\varepsilon \subset V$  and a smooth function  $\tau^\varepsilon : V^\varepsilon \rightarrow \mathbb{R}$  with the following properties:*

- (a)  $X\tau^\varepsilon = 1$ .
- (b)  $|d_q \tau^\varepsilon(v)| \leq A\varepsilon^\theta \|v\|$ , for all  $q \in V^\varepsilon$  and  $v \in E^{ss}$ , where  $A$  is a constant independent of  $\varepsilon$ ,  $q$ , and  $v$ .
- (c)  $\|d\tau^\varepsilon\| \leq B\varepsilon^{\theta-1}$ , where  $B$  is a constant independent of  $\varepsilon$ .
- (d) The Hausdorff distance between  $V$  and  $V^\varepsilon$  tends to zero, as  $\varepsilon \rightarrow 0$ .

*Proof.* For the sake of notational simplicity, we will write  $f(x) \lesssim g(x)$  ( $x \in S$ ) to mean that there exists a constant  $C$  independent of  $x \in S$  such that  $f(x) \leq Cg(x)$ , for all  $x \in S$ .

Since  $U$  is contained in a smooth chart for  $M$ , by Corollary 2.2 there exists  $\varepsilon_*$  such that for each  $i > n_s$  there is a family  $\varphi_i^\varepsilon$  ( $0 < \varepsilon < \varepsilon_*$ ) of smooth approximations of  $\varphi_i$  satisfying

$$\|\varphi_i^\varepsilon - \varphi_i\|_{C^0} \lesssim \varepsilon^\theta \quad \text{and} \quad \|d\varphi_i^\varepsilon\|_{C^0} \lesssim \varepsilon^{\theta-1}. \quad (2.2)$$

Note that  $u^\varepsilon$  is defined on all of  $U$ . Denote by  $\mathcal{F}_\varepsilon$  the foliation of  $U^\varepsilon$  defined by

$$\varphi_{n_s+1}^\varepsilon = \text{constant}, \dots, \varphi_n^\varepsilon = \text{constant}.$$

It is easy to see that  $\mathcal{F}_\varepsilon$  is smooth and the (largest principal) angle between  $W_{\text{loc}}^{ss}$  and  $\mathcal{F}_\varepsilon$  is  $\lesssim \varepsilon^\theta$ . Let

$$\Sigma^\varepsilon = \bigcup_{x \in W_{\text{loc}}^{uu}(p)} \mathcal{F}_\varepsilon(x).$$

Then  $\Sigma^\varepsilon$  is a smooth local cross section and there exists  $T_\varepsilon > 0$  such that the set

$$V^\varepsilon = \bigcup_{|t| < T_\varepsilon} f_t(\Sigma^\varepsilon)$$

is a smooth flow box for  $X$  contained in  $U^\varepsilon$ . It is clear that the length  $T_\varepsilon$  of  $V^\varepsilon$  is close to the length  $T$  of  $V$ ; we can also take  $T_\varepsilon \leq T$ .

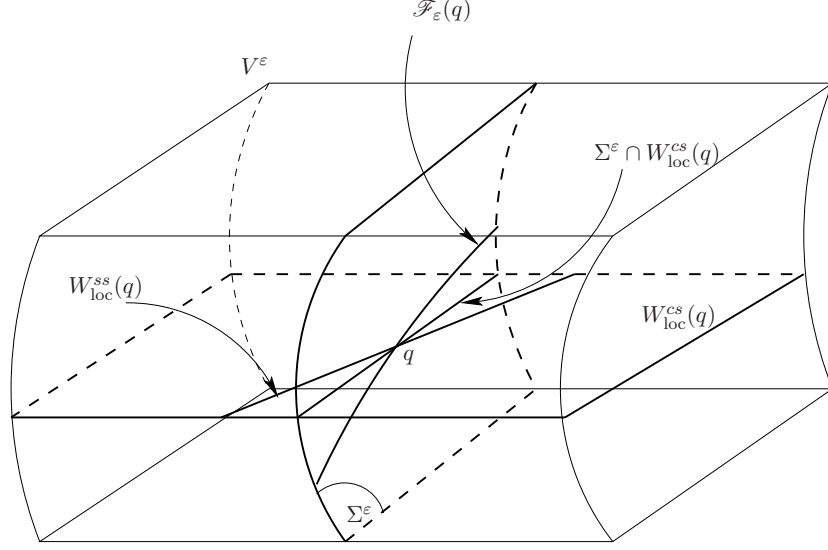
Define  $\tau^\varepsilon : V^\varepsilon \rightarrow \mathbb{R}$  by

$$\tau^\varepsilon(f_t q) = t,$$

for all  $q \in \Sigma^\varepsilon$  and  $-T_\varepsilon < t < T_\varepsilon$ . Clearly,  $\tau^\varepsilon$  is smooth and  $X\tau^\varepsilon = 1$ , proving (a).

We will first show that (b) holds along  $\Sigma^\varepsilon$ . Let  $q \in \Sigma^\varepsilon$  be arbitrary. Since the angle between  $\mathcal{F}_\varepsilon(q)$  and  $W_{\text{loc}}^{ss}(q)$  is  $\lesssim \varepsilon^\theta$ , it follows that the angle between  $\Sigma^\varepsilon \cap W_{\text{loc}}^{cs}(q)$  and  $W_{\text{loc}}^{ss}(q)$  is also  $\lesssim \varepsilon^\theta$ . See Figure 1. Then the fact that  $d\tau^\varepsilon = 0$  on  $\Sigma^\varepsilon \cap W_{\text{loc}}^{cs}(q)$  implies  $\|d_q \tau^\varepsilon \upharpoonright_{E^{ss}}\| \lesssim \varepsilon^\theta$ , as desired. To extend this to all points in  $V^\varepsilon$  we will use the invariance of  $d\tau^\varepsilon$  with respect to the flow:  $f_t^*(d\tau^\varepsilon) = d\tau^\varepsilon$ , whenever both sides are defined. For  $q \in \Sigma^\varepsilon$ ,  $|t| < T_\varepsilon$  and  $v \in E_{f_t q}^{ss}$ , we have

$$\begin{aligned} |d_{f_t q} \tau^\varepsilon(v)| &= |d_q \tau^\varepsilon(Tf_{-t}(v))| \\ &\leq \|d_q \tau^\varepsilon \upharpoonright_{E^{ss}}\| \|Tf_{-t}(v)\| \\ &\lesssim \varepsilon^\theta \mu_-^{-t} \|v\| \\ &\leq \varepsilon^\theta \mu_-^{-T_\varepsilon} \|v\| \\ &\leq \varepsilon^\theta \mu_-^{-T} \|v\|, \end{aligned}$$

FIGURE 1. The flow box  $V^\varepsilon$  with local cross section  $\Sigma^\varepsilon$ .

where we used  $T_\varepsilon \leq T$ . This proves (b) for all  $q \in V^\varepsilon$ .

To prove (c), we work in a smooth coordinate system in which  $X = \partial/\partial x_1$ . Since  $\Sigma$  is a  $C^\theta$  hypersurface, it is locally the graph of a  $C^\theta$  function  $g$  such that  $\tau(g(z), z) = 0$ , for all  $z$  is some open set in  $\mathbb{R}^{n-1}$ . Similarly,  $\Sigma^\varepsilon$  is locally the graph of a smooth function  $g^\varepsilon$  such that  $\tau^\varepsilon(g^\varepsilon(z), z) = 0$ , for all  $z$  in some open set in  $\mathbb{R}^{n-1}$ . Since  $g^\varepsilon \rightarrow g$ , as  $\varepsilon \rightarrow 0$ , (2.2) forces  $\|dg^\varepsilon\| \lesssim \varepsilon^{\theta-1}$ . Differentiating  $\tau^\varepsilon(g^\varepsilon(z), z) = 0$  with respect to  $z_i$  for  $i > 1$  and using  $\partial\tau^\varepsilon/\partial z_1 = X\tau^\varepsilon = 1$ , we obtain

$$\left| \frac{\partial\tau^\varepsilon}{\partial z_i} \right| = \left| \frac{\partial g^\varepsilon}{\partial z_i} \right| \lesssim \varepsilon^{\theta-1}.$$

Thus (c) holds on  $\Sigma^\varepsilon$ ; we can extend it to  $V^\varepsilon$  using the flow invariance of  $d\tau^\varepsilon$  as in the proof of (b).

Part (d) holds by construction.  $\square$

By the above analysis and compactness, we can cover  $M$  by finitely many Hölder flow boxes  $V_1, \dots, V_\ell$ , each of which is equipped with a local Hölder cross section  $\Sigma_i$ . We can approximate each  $\Sigma_i$  by a smooth cross section  $\Sigma_i^\varepsilon$  as above and obtain smooth flow boxes  $V_i^\varepsilon \subset V_i$  and smooth functions  $\tau_i^\varepsilon : V_i^\varepsilon \rightarrow \mathbb{R}$  satisfying the properties from Lemma 2.3; namely,

$$X\tau_i^\varepsilon = 1, \quad \|d\tau_i^\varepsilon|_{E^{ss}}\| \leq A_i\varepsilon^\theta, \quad \|d\tau_i^\varepsilon\| \leq B_i\varepsilon^{\theta-1}, \quad (2.3)$$

where the constants  $A_i, B_i$  are independent of  $\varepsilon$ . Let  $A = \max A_i, B = \max B_i$  (not to be confused with the constants  $A, B$  in Lemma 2.3). By Lemma 2.3 (d), there exists  $\varepsilon_* > 0$  such that for all  $0 < \varepsilon < \varepsilon_*$  the sets  $V_1^\varepsilon, \dots, V_\ell^\varepsilon$  cover  $M$ .

**2.3. Distance to the space of closed forms.** We will consider  $C^r$  differential  $k$ -forms  $\xi$ , with  $r \geq 1$ . We denote the  $C^0$  norm of  $\xi$  on  $M$  by  $\|\xi\|$ :

$$\|\xi\| = \sup_{p \in M} |\xi_p|, \quad (2.4)$$

where  $|\xi_p|$  is the operator norm of  $\xi_p$  as a  $k$ -linear map  $T_p M \times \dots \times T_p M \rightarrow \mathbb{R}$ .

Consider first a differential form  $\omega$  on  $M \times [0, 1]$ , where  $t$  is the coordinate in  $[0, 1]$ . Denote by  $\pi_M : M \times [0, 1] \rightarrow M$  and  $\pi_I : M \times [0, 1] \rightarrow [0, 1]$  the obvious projections. Since

$$T_{(p,t)}(M \times [0, 1]) = T_p M \oplus T_t[0, 1],$$

any differential  $k$ -form on  $M \times [0, 1]$  can be uniquely written as

$$\omega = \omega_0 + dt \wedge \eta,$$

where  $\omega_0(v_1, \dots, v_k) = 0$  if some  $v_i$  is in the kernel of  $(\pi_M)_*$  and  $\eta$  is a  $(k-1)$ -form with the analogous property (i.e.,  $i_v \omega_0 = i_v \eta = 0$ , for every “vertical” vector  $v \in T(M \times [0, 1])$ , where  $i_v$  denotes contraction by  $v$ ).

Define a  $(k-1)$ -form  $\mathcal{H}(\xi)$  on  $M$  by

$$\mathcal{H}(\omega)_p = \int_0^1 j_t^* \eta_{(p,t)} dt,$$

where  $j_t : M \rightarrow M \times [0, 1]$  is defined by  $j_t(p) = (p, t)$ . It is well-known (cf., [Spi05]) that

$$j_1^* \omega - j_0^* \omega = d(\mathcal{H}\omega) + \mathcal{H}(d\omega). \quad (2.5)$$

Considering  $\mathcal{H}$  as a linear operator from  $\Omega^k(M \times [0, 1])$  to  $\Omega^{k-1}(M)$ , both equipped with the  $C^0$  norm, it is not hard to see that

$$\|\mathcal{H}\| = 1. \quad (2.6)$$

We claim:

**2.4. Proposition.** *Let  $\xi$  be a  $C^r$  differential  $k$ -form ( $r \geq 1$ ) on a closed manifold  $M$ . Then*

$$\inf\{\|\xi - \eta\| : \eta \in C^r, d\eta = 0\} \leq \|d\xi\|.$$

In other words,  $\|d\xi\|$  is an upper bound on the distance from  $\xi$  to the space of closed forms. The inequality also holds for continuous forms which admit a continuous exterior derivative.

*Proof of the Proposition.* First, let us show that the result holds on any manifold  $M$  which is smoothly contractible to a point  $p_0$  via  $H : M \times [0, 1] \rightarrow M$ , where  $H(p, 0) = p_0$  and  $H(p, 1) = p$ , for all  $p \in M$ . Since  $H \circ j_1$  is the identity map of  $M$  and  $H \circ j_0$  is the constant map  $p_0$ , it follows that

$$\xi = (H \circ j_1)^* \xi = j_1^*(H^* \xi) \quad \text{and} \quad 0 = (H \circ j_0)^* \xi = j_0^*(H^* \xi).$$

Applying (2.5) to  $H^* \xi$ , we obtain

$$\begin{aligned} \xi &= \xi - 0 \\ &= j_1^*(H^* \xi) - j_0^*(H^* \xi) \\ &= d\mathcal{H}(\xi) + \mathcal{H}(d\xi). \end{aligned}$$

Using (2.6), we obtain

$$\|\xi - d\mathcal{H}(\xi)\| = \|\mathcal{H}(d\xi)\| \leq \|d\xi\|.$$

Therefore, the statement of the theorem holds for contractible  $M$ .

Let  $M$  now be any closed manifold and  $\xi$  a  $C^r$   $k$ -form on  $M$ ,  $r \geq 1$ . Cover  $M$  by contractible open sets  $U_1, \dots, U_m$ . Denote the operator  $\mathcal{H}$  restricted to forms on  $U_i$  by  $\mathcal{H}_i$  and let  $\xi_i$  be the restriction of  $\xi$  to  $U_i$ . Define a  $k$ -form  $\eta$  on  $M$  by requiring that the restriction of  $\eta$  to  $U_i$  be equal to  $d\mathcal{H}_i(\xi_i)$ . We claim that  $\eta$  is well-defined and closed.



Indeed,  $\xi_i = \xi_j$  on  $U_i \cap U_j$ , and  $\mathcal{H}_i(\omega) = \mathcal{H}_j(\omega)$  for every  $k$ -form  $\omega$  defined on  $(U_i \cap U_j) \times [0, 1]$ . Thus on  $U_i \cap U_j$ , we have  $d\mathcal{H}_i(\xi_i) = d\mathcal{H}_j(\xi_j)$ , so  $\eta$  is well-defined. Since  $\eta$  is locally exact, it follows that it is closed. By (2.4), we obtain

$$\|\xi - \eta\| \leq \max_{1 \leq i \leq m} \|\xi_i - d\mathcal{H}_i(\xi_i)\| \leq \max_{1 \leq i \leq m} \|d\xi_i\| \leq \|d\xi\|.$$

This completes the proof of the proposition.  $\square$

**2.4. Change of Riemannian metric.** Denote by  $\Omega$  the smooth volume form preserved by the flow and let  $\mathcal{R}$  be the Riemannian metric which induces  $\Omega$ .

Our goal is to show that relative to some Riemannian metric the area of the parallelogram  $Tf_{-t}(v \wedge w)$  grows as  $(\mu_+^{n_s-1} \lambda_+^{n_u-1})^t$ , where  $v \in E^{ss}$ ,  $w \in E^{uu}$ , and  $t \geq 0$ . To do this, it will be convenient to switch from the original Riemannian metric  $\mathcal{R}$  to a new metric  $\mathcal{R}'$  with respect to which  $X$  is a unit vector and  $E^c \oplus E^{ss} \oplus E^{uu}$  is an orthogonal splitting. This metric can in general be only continuous and the corresponding volume form  $\Omega'$  may not be invariant with respect to the flow. We will show that this does not present a problem.

Let  $\mathcal{R}'$  be as above and let  $\Omega'$  be the Riemannian volume form induced by  $\mathcal{R}'$ . Since  $\Omega$  and  $\Omega'$  are both volume forms, there exists a positive continuous function  $\phi$  such that  $\Omega' = \phi \Omega$ . Let

$$L = \frac{\max_M \phi}{\min_M \phi}.$$

Denote the norms of tangent vectors (and their wedge products) with respect to  $\mathcal{R}$  and  $\mathcal{R}'$  by  $\|\cdot\|$  and  $\|\cdot\|'$ , respectively. By compactness of  $M$  there exist  $b_-, b_+ > 0$  such that

$$b_- \|v\| \leq \|v\|' \leq b_+ \|v\|,$$

for all  $v \in TM$ . Observe that for  $v \in E^{ss}$ ,  $w \in E^{uu}$ , and  $t \geq 0$ , we have

$$\|Tf_t(v)\|' \leq bc\mu_+^t \|v\|' \quad \text{and} \quad \|Tf_t(w)\|' \leq bc\lambda_+^t \|w\|',$$

where  $b = b_+/b_-$ . It is easy to check that

$$f_t^* \Omega' = \frac{\phi \circ f_t}{\phi} \Omega',$$

for all  $t \in \mathbb{R}$ . Thus for any  $n$ -dimensional parallelepiped  $\Pi$  in a tangent space to  $M$  and  $t \in \mathbb{R}$ , we have

$$\|\Pi\|' \leq L \|Tf_t(\Pi)\|'$$

**2.5. Lemma.** *If  $\Phi = \{f_t\}$  is a volume preserving Anosov flow with constants defined in (1.1) and (1.2), then*

$$\|Tf_{-t}(v \wedge w)\|' \leq L(bc)^{n-3} (\mu_+^{n_s-1} \lambda_+^{n_u-1})^t \|v \wedge w\|',$$

for all  $v \in E^{ss}$ ,  $w \in E^{uu}$  and  $t \geq 0$ .

*Proof.* Let  $v \in E^{ss}$ ,  $w \in E^{uu}$  and  $t \geq 0$  be arbitrary. Set  $v_1 = Tf_{-t}(v)$  and  $w_1 = Tf_{-t}(w)$ . Choose vectors  $v_2, \dots, v_{n_s} \in E^{ss}$  and  $w_2, \dots, w_{n_u} \in E^{uu}$  such that, relative to  $\mathcal{R}'$ ,  $(X, v_1, \dots, v_{n_s}, w_1, \dots, w_{n_u})$  is an orthogonal basis of the corresponding tangent space and  $v_i, w_i$  are all of unit length, for  $i \geq 2$ .

Then:

$$\begin{aligned}
\|Tf_{-t}(v \wedge w)\|' &= \|v_1 \wedge w_1\|' \\
&= \|X \wedge v_1 \wedge \cdots \wedge v_{n_s} \wedge w_1 \wedge \cdots \wedge w_{n_u}\|' \\
&= \|\Pi\|' \\
&\leq L \|Tf_t(\Pi)\|' \\
&= L \|Tf_t(X \wedge v_1 \wedge \cdots \wedge v_{n_s} \wedge w_1 \wedge \cdots \wedge w_{n_u})\|' \\
&\leq L \|Tf_t(v_1 \wedge w_1)\|' \|Tf_t(X \wedge v_2 \wedge \cdots \wedge v_{n_s} \wedge w_2 \wedge \cdots \wedge w_{n_u})\|' \\
&\leq L \|v \wedge w\|' (bc)^{n_s-1} \mu_+^{(n_s-1)t} \cdot (bc)^{n_u-1} \lambda_+^{(n_u-1)t} \\
&= L (bc)^{n-3} \mu_+^{(n_s-1)t} \lambda_+^{(n_u-1)t} \|v \wedge w\|. \quad \square
\end{aligned}$$

In the remainder of the paper we will always be working with  $\mathcal{R}'$  as the underlying Riemannian metric on  $M$ ; the norms of tangent vectors and differential forms are taken relative to  $\mathcal{R}'$  and will be denoted by the symbol  $\|\cdot\|$  (thus slightly abusing notation for the sake of keeping it less cumbersome).

### 3. PROOF OF THE MAIN THEOREM

We will construct a smooth closed 1-form  $\eta$  such that  $u = \eta(X) > 0$ . Assuming for a moment that such a form has been found, the proof can be completed as follows. Define

$$\tilde{X} = \frac{1}{u} X.$$

Then  $\tilde{X}$  is an Anosov vector field [AS67] and  $\eta(\tilde{X}) = 1$ . Thus the Lie derivative of  $\eta$  with respect to  $\tilde{X}$  satisfies

$$L_{\tilde{X}}\eta = (di_{\tilde{X}} + i_{\tilde{X}}d)\eta = 0,$$

which implies that  $\eta$  is invariant with respect to the flow. It follows that its kernel  $\text{Ker}(\eta)$  is an invariant codimension one distribution transverse to the flow, so  $\text{Ker}(\eta)$  is forced to be the sum  $\tilde{E}^{su} = \tilde{E}^{ss} \oplus \tilde{E}^{uu}$  of the strong stable and strong unstable bundles of  $\tilde{X}$ . Since  $\eta$  is closed,  $\tilde{E}^{su}$  is uniquely integrable, so by Plante [Pla72],  $\tilde{X}$  admits a global cross section  $\Sigma$ , which is also a global cross section for  $X$ .

So it remains to construct a smooth 1-form  $\eta$  with  $\eta(X) > 0$ , which will be done in three steps. In the first two steps we construct a smooth 1-form  $\xi$  such that

$$\xi(X) = 1 \quad \text{and} \quad \|d\xi\| < 1.$$

The third step consists of approximating  $\xi$  by a smooth closed 1-form using Proposition 2.4.

**Step 1.** Let  $\varepsilon > 0$  be arbitrary. As in §2.2, for  $\varepsilon < \varepsilon_*$ , we can cover  $M$  by smooth flow boxes  $V_1^\varepsilon, \dots, V_\ell^\varepsilon$ , with  $\ell$  independent of  $\varepsilon$ , such that with respect to the Hausdorff distance each  $V_i^\varepsilon$  is close to a fixed open set  $V_i$ . In addition, we have smooth functions  $\tau_i^\varepsilon : V_i^\varepsilon \rightarrow \mathbb{R}$  such that

$$X\tau_i^\varepsilon = 1, \quad \|d\tau_i^\varepsilon|_{E^{ss}}\| \leq A\varepsilon^\theta, \quad \text{and} \quad \|d\tau_i^\varepsilon\| \leq B\varepsilon^{\theta-1}, \quad (3.1)$$

where  $A, B$  are constants independent of  $\varepsilon$ .

Let  $\{\psi_i^\varepsilon\}$  be a smooth partition of unity subordinate to the cover  $\{V_i^\varepsilon\}$ . Since  $\ell$  and the sizes of the sets  $V_i^\varepsilon$  are independent of  $\varepsilon$ , there is a constant  $C > 0$  also independent of  $\varepsilon$  such that

$$\sum_{i=1}^{\ell} \|d\psi_i^\varepsilon\| \leq C, \quad (3.2)$$

for all  $0 < \varepsilon < \varepsilon_*$ . Define

$$\xi_0^\varepsilon = \sum_{i=1}^{\ell} \psi_i^\varepsilon d\tau_i^\varepsilon.$$

**3.1. Lemma.**  $\xi_0^\varepsilon$  is a smooth 1-form on  $M$  with the following properties:

- (a)  $\xi_0^\varepsilon(X) = 1$ ;
- (b)  $\|\xi_0^\varepsilon|_{E^{ss}}\| \leq A\varepsilon^\theta$ , for all  $0 < \varepsilon < \varepsilon_*$ ;
- (c)  $|d\xi_0^\varepsilon(v, w)| \leq D\varepsilon^\theta$ , for all unit vectors  $v, w \in E^{cs}$  and  $0 < \varepsilon < \varepsilon_*$ , where  $D$  is a constant independent of  $\varepsilon$ ;
- (d)  $\|d\xi_0^\varepsilon\| \leq K\varepsilon^{\theta-1}$ , for every  $0 < \varepsilon < \varepsilon_*$ , where  $K$  is a constant independent of  $\varepsilon$ .

*Proof.* Part (a) is clear. Part (b) follows easily from the second inequality in (3.1). To prove (c), first note that if  $\beta$  is a bilinear form on an inner product space  $E$  which splits into two orthogonal subspaces  $E_1$  and  $E_2$  and  $\beta_{ij} = \beta|_{E_i \times E_j}$ , then

$$\|\beta\| \leq \|\beta_{11}\| + 2\|\beta_{12}\| + \|\beta_{22}\|. \quad (3.3)$$

We fix  $p \in M$  and take  $\beta = d_p \xi_0^\varepsilon$ ,  $E_1 = E_p^c$  and  $E_2 = E_p^{ss}$ . Since

$$d\xi_0^\varepsilon = \sum_{i=1}^{\ell} d\psi_i^\varepsilon \wedge d\tau_i^\varepsilon,$$

if  $v, w \in E^{ss}$  are unit vectors, then

$$|d\xi_0^\varepsilon(v, w)| \leq 2C\varepsilon^\theta,$$

and

$$|d\xi_0^\varepsilon(X, v)| = \left| \sum_i d\psi_i^\varepsilon(X) d\tau_i^\varepsilon(v) - d\psi_i^\varepsilon(v) d\tau_i^\varepsilon(X) \right| = \left| \sum_i d\psi_i^\varepsilon(X) d\tau_i^\varepsilon(v) \right| \leq AC\varepsilon^\theta,$$

where we used  $d\tau_i^\varepsilon(X) = 1$  and  $\sum_i d\psi_i^\varepsilon(v) = d(\sum_i \psi_i^\varepsilon)(v) = 0$ . Thus by (3.3) we can take  $D = (A + 2)C\varepsilon$  in (c). Finally, (3.2) and  $\|d\tau_i^\varepsilon\| \leq B\varepsilon^{\theta-1}$  imply

$$\|d\xi_0^\varepsilon\| = \left\| \sum_{i=1}^{\ell} d\psi_i^\varepsilon \wedge d\tau_i^\varepsilon \right\| \leq BC\varepsilon^{\theta-1},$$

so (d) holds with  $K = BC$ . □

In summary, for every  $0 < \varepsilon < \varepsilon_*$  we have a 1-form which is small on  $E^{ss}$ , whose exterior derivative is small when restricted to  $E^{cs}$ , but whose overall norm grows as  $\varepsilon^{\theta-1}$ , as  $\varepsilon \rightarrow 0$ .

**Step 2.** To remedy the problem represented by part (d) of the previous lemma, we flow backwards and use Lemma 2.5. Let  $t > 0$  be large (how large will be specified shortly) and set

$$\xi_t^\varepsilon = f_{-t}^* \xi_0^\varepsilon.$$

**3.2. Lemma.** *There exists a constant  $H > 0$  independent of  $\varepsilon$  and  $t$  such that*

$$\|d\xi_t^\varepsilon\| \leq \max \left\{ H\varepsilon^{\theta-1}(\mu_+^{n_s-1}\lambda_+^{n_u-1})^t, H\varepsilon^\theta\mu_-^{-2t} \right\}$$

*Proof.* Observe that if  $\beta$  is a bilinear form as in the proof of the previous lemma and  $\|\beta_{22}\| \leq \|\beta_{ij}\|$ , for all  $i, j = 1, 2$ , then

$$\|\beta\| \leq 4 \max(\|\beta_{11}\|, \|\beta_{12}\|).$$

Fix  $p \in M$  and take  $\beta = d_p \xi_t^\varepsilon$ ,  $E = T_p M$ ,  $E_1 = E_p^{cs}$  and  $E_2 = E_p^{uu}$ . Since  $d\xi_t^\varepsilon = f_{-t}^*(d\xi_0^\varepsilon)$  and  $\|Tf_{-t}|_{E^{cs}}\| \leq c\mu_-^{-t}$ , for  $t > 0$ , Lemmas 3.1 and 2.5 imply

$$\|\beta_{11}\| \leq c^2 D \varepsilon^\theta \mu_-^{-2t} \quad \text{and} \quad \|\beta_{12}\| \leq KL \varepsilon^{\theta-1} (bc)^{n-3} (\mu_+^{n_s-1} \lambda_+^{n_u-1})^t.$$

Note that the second inequality holds because if  $v \in E^{cs}$  and  $w \in E^{uu}$ , then  $|d\xi_t^\varepsilon(v, w)| = |d\xi_0^\varepsilon(T_{-t}(v \wedge w))|$  is largest if  $v \in E^{ss}$ . Observe also that  $\|\beta_{22}\|$  is smaller than  $\|\beta_{ij}\|$  ( $i, j = 1, 2$ ) since the flow contracts strong unstable vectors in negative time.

The proof of the lemma is now complete with  $H = 4 \max(c^2 D, KL(bc)^{n-3})$ .  $\square$

Since  $\xi_t^\varepsilon(X) = 1$ , for all  $0 < \varepsilon < \varepsilon_*$  and  $t > 0$ , the following lemma will complete the construction of the desired form  $\xi$ .

**3.3. Lemma.** *There exist  $0 < \varepsilon < \varepsilon_*$  and  $t > 0$  such that  $\|d\xi_t^\varepsilon\| < 1$ .*

*Proof.* By the previous Lemma, it suffices to show that the following system of inequalities

$$\begin{aligned} H\varepsilon^{\theta-1}(\mu_+^{n_s-1}\lambda_+^{n_u-1})^t &< 1 \\ H\varepsilon^\theta\mu_-^{-2t} &< 1 \end{aligned}$$

admits a solution  $\varepsilon, t > 0$ . Solving the first inequality for  $t$  we obtain

$$t > \frac{(1-\theta)\log\varepsilon - \log H}{(n_s-1)\log\mu_+ + (n_u-1)\log\lambda_+}. \quad (3.4)$$

The second inequality is equivalent to

$$t < \frac{\log H + \theta \log \varepsilon}{2 \log \mu_-}. \quad (3.5)$$

There exists  $t$  with these properties if and only if we can find  $0 < \varepsilon < \varepsilon_*$  such that

$$\frac{(1-\theta)\log\varepsilon - \log H}{(n_s-1)\log\mu_+ + (n_u-1)\log\lambda_+} < \frac{\log H + \theta \log \varepsilon}{2 \log \mu_-}.$$

Since  $H$  is fixed and  $\varepsilon$  is small, this is possible if

$$\frac{(1-\theta)\log\varepsilon}{(n_s-1)\log\mu_+ + (n_u-1)\log\lambda_+} < \frac{\theta \log \varepsilon}{2 \log \mu_-},$$

which is equivalent to (1.3).  $\square$

**Step 3.** Choose  $0 < \varepsilon < \varepsilon_*$  and  $t > 0$  such that  $\|d\xi_t^\varepsilon\| < 1$  and set

$$\xi = \xi_t^\varepsilon.$$

By Proposition 2.4 there exists a smooth closed 1-form  $\eta$  such that

$$\|\xi - \eta\| \leq \|d\xi\| < 1.$$

Since  $\xi(X) = 1$ , it follows that  $\eta(X) > 0$ . This completes the construction of  $\eta$  and concludes the proof of the theorem.  $\square$

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